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# NUMERICAL SIMULATION OF THE PROPAGATION OF A SOLITARY WAVE OVER UNDERWATER BARRIERS

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*The problem on the behavior of a solitary wave passing over different types of underwater barriers is considered. The problem has been solved numerically using the method of the correction of flows. The parameters of the method of solution have been selected. The results are compared with the Korteweg–de Vries theory. The interaction of two solitary waves is simulated and it is shown that they possess the properties of solitons. After the interaction, their shape and velocity remain constant; it is only the phase that undergoes a change. The case of disintegration of the initial perturbation in a cylindrically symmetric problem is considered.*

**Introduction.** Study of the propagation of waves on the surface of water is a classical problem, and the interest shown in it is due to the necessity of solving a variety of applied problems [1]. We consider gravitational waves, because the main physical factor that influences the propagation of long waves on water is the gravitational force. Two types of waves determined by the ratio between the depth of the basin and the wave length are distinguished. They correspond to deep and shallow water. The deep water gives rise to waves whose length is smaller than the depth of the basin, while the shallow water gives rise to waves whose length is much in greater than the basin depth. In the shallow water-approximation it is assumed that the entire layer of the liquid moves, whereas for the deep water it is assumed that only the surface layers of water are perturbed.

Under certain conditions, e.g., in the case of an underwater earthquake, the arising water may sometimes lead to a tsunami having a large length (up to 100 km) and a small height, and therefore even the deepest places in the ocean may be shallow for them. Under initial conditions of a certain type, a tsunami is likely to be considered as a solitary wave (soliton). Obviously, the bottom relief will influence various characteristics of the wave, in particular, the energy ones. This problem is of particular interest because the damage that may be incurred depends on the energy of waves. For this reason, the problem of studying the influence exerted by underwater barriers on the propagation of waves in water is an urgent one. In the present work, we consider the problem of simulation of gravitational waves in the "shallow water" approximation [2] and investigate the influence of the bottom relief on the propagation of waves.

**Statement of the Problem.** Let there be a basin with water of depth $h(x)$ and length $L$. We assume that a perturbation in the form of a stationary solitary wave is assigned at the time zero.

The system of equations describing the problem is used in the form [3]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial z}{\partial x} = 0\,, \quad \frac{\partial}{\partial x}(u\,(h+z)) + \frac{\partial z}{\partial t} = 0\,. \tag{1}$$

The first equation of system (1) characterizes the motion of the liquid in the gravity field; the second is the continuity equation. Both have been obtained in the "shallow water" approximation. This means that [2] $a \ll h$ and $h \ll L$.

The initial and boundary conditions are

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$$z\,(x,0) = 2k^2 \cosh^{-2}(kx) + h_0\,, \quad u\,(x,0) = 0\,, \quad z_x\,(0,t) = 0\,, \quad u\,(0,t) = 0\,,$$
$$0 \le x \le L\,, \quad 0 \le t \le T\,. \tag{2}$$

We assume that at the initial instant of time the velocity of the perturbation is zero; with time the perturbation is decomposed into two symmetric waves moving in opposite directions. To simplify the solution of the problem, we will consider only the wave that moves to the right. Therefore, we will use symmetric conditions at the left boundary and zero at the right boundary. As the initial profile we take that of a doubled soliton in order that the wave formed could be compared with the solitary wave described by the Korteweg–de Vries equation [2].

The system yielding the Korteweg–de Vries equation has the form [2]

$$z_t + (uz)_x = 0\,, \quad u_t + uu_x + gz_x + \frac{1}{3}c_0^2 h_{xxx} = 0\,. \tag{3}$$

We note that system (3) differs from (1) only in the presence of the dispersion term $\frac{1}{3}c_0^2 h_{xxx}$, precisely which is responsible for the possibility of the soliton solution. A soliton on water was seen by Russell in 1834 when he tested a barge in a channel. Using (3), Korteweg and de Vries in 1895 derived an equation that corresponded to Russell's experiments in the "shallow" water approximation.

**Difference Scheme.** For numerical solution of problem (1) with initial and boundary conditions in the form of (2) we will use the method of correction of flows [4]. For this purpose, we will introduce the grid function $z_j^i$, where the subscript denotes the number of the point and the superscript the number of the layer in time. To simplify the representation for the current temporal layer, we assume that $i = 0$ and for the next one that $i = 1$. At the first layer the function is unknown and it should be calculated. We also introduce an intermediate layer, $\tilde{z}_j$, needed for determining the antidiffusion flows $\{\tilde{\Phi}_{j+1/2}\}$. Similar grid functions $u_j^i$ are used for velocity.

The difference scheme, as applied to the second equation of system (1), has the form

$$\tilde{z}_j = z_j^0 - \frac{1}{2}[\varepsilon_{j+1/2}(z_{j+1}^0 + z_j^0) - \varepsilon_{j-1/2}(z_j^0 + z_{j-1}^0)] + [\nu_{j+1/2}(z_{j+1}^0 - z_j^0) -$$
$$- \nu_{j-1/2}(z_j^0 - z_{j-1}^0)] - \frac{\Delta t}{2\Delta x}(u_{j+1}^0 h_{j+1} - u_{j-1}^0 h_{j-1})\,, \tag{4}$$

$$\varepsilon_{j+1/2} = u_{j+1/2}\frac{\Delta t}{\Delta x}\,, \quad \nu_{j+1/2} = \mu + \frac{1}{2}\varepsilon_{j+1/2}^2\,, \quad \mu = \text{const}\,; \tag{5}$$

$$\Phi_{j+1/2} = \mu\,(\tilde{z}_{j+1} - \tilde{z}_j)\,, \tag{6}$$

$$\tilde{\Phi}_{j+1/2} = s \max\,[0, \min\,[S\,(\tilde{z}_{j+2} - \tilde{z}_{j+1})\,, \ |\Phi_{j+1/2}|\,, \ s\,(\tilde{z}_j - \tilde{z}_{j-1})]]\,; \tag{7}$$

$$\Phi_{j-1/2} = \mu\,(\tilde{z}_j - \tilde{z}_{j-1})\,; \tag{8}$$

$$\tilde{\Phi}_{j-1/2} = s \max\,[0, \min\,[S\,(\tilde{z}_{j+1} - \tilde{z}_j)\,, \ |\Phi_{j-1/2}|\,, \ s\,(\tilde{z}_{j-1} - \tilde{z}_{j-2})]]\,; \tag{9}$$

$$s = \text{sign}\,(\tilde{z}_{j+1} - \tilde{z}_j)\,, \quad z_j^1 = \tilde{z}_j - \tilde{\Phi}_{j+1/2} + \tilde{\Phi}_{j-1/2}\,. \tag{10}$$

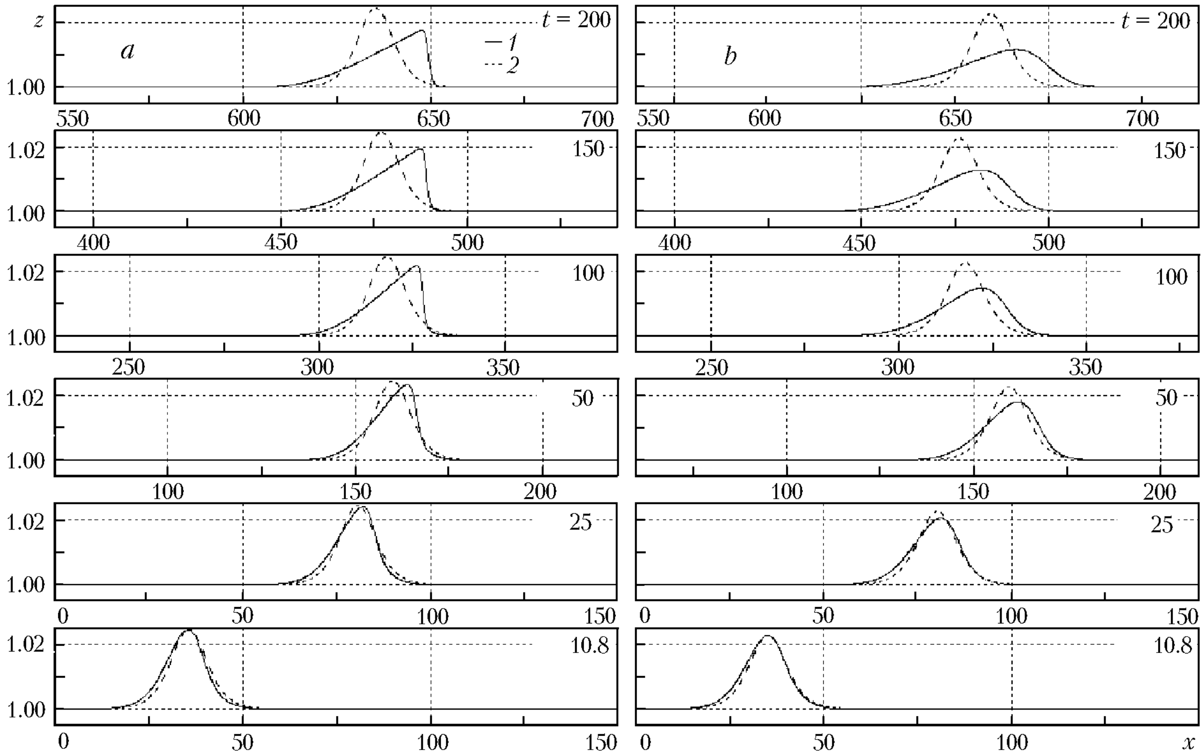


Fig. 1. Change in the wave profile with time for various antidiffusion coefficients $\mu$ = 0.39 (a) and 0.49 (b): 1) profile obtained by numerical solution; 2) soliton obtained by solving the Korteweg–de Vries equation. $t$, sec; $z$, $x$, m.

The difference equation (4) corresponds to the second term of system (1) written in a conservative form to ensure the conservation of mass. It involves the terms with dimensionless coefficients of diffusion $\left\{\nu_{j+1/2}\right\}$ to ensure the condition of positiveness since the water level cannot become negative. To decrease the influence of diffusion, explicit antidiffusion with two-sided correction is used. The antidiffusion flows are calculated from Eqs. (6) and (8). Their correction is made according to Eqs. (7) and (9). The correction is needed in order to prevent the formation of new minima in the profile at the stage of antidiffusion and the decrease of those available. The same is true of the maxima. Otherwise, the antidiffusion will introduce nonphysical changes into the wave profile, which will distort the results. For example, a decrease in the value of a minimum in the solution may cause the appearance of a negative value.

We note that this scheme involves the coefficient of antidiffusion $\mu$, which has no physical meaning and was introduced into the difference scheme only to raise the stability of the numerical solution. The coefficient is also a term in the diffusion coefficients $\left\{\nu_{j+1/2}\right\}$. Therefore, the best results can be obtained at the minimum possible value of this coefficient. The difference scheme is similar for the first term of system (1).

To study the numerical solution of the propagation of a wave, we made comparison with a solitary wave described by the solution of the Korteweg–de Vries equation [2]. For this purpose, the wave formed was interpolated by a soliton moving in the same direction at a constant velocity prescribed depending on the height of the soliton and the depth of the bottom, according to [2].

**Selection of the Antidiffusion Parameter.** To select the value of $\mu$, we performed simulation of the wave motion over a plane bottom.

Since the dimensionless coefficient of antidiffusion has no physical meaning, it can be found from a numerical experiment.

For this purpose, we performed a series of numerical calculations which showed that in the grid of size $h$ = 0.05 and $\tau$ = 0.005 the antidiffusion coefficient can take on values from 0.39 to 0.50. Moreover, simulation on other grids showed that the upper limit of this range is fixed, whereas the lower depends on the dimensions of the grid

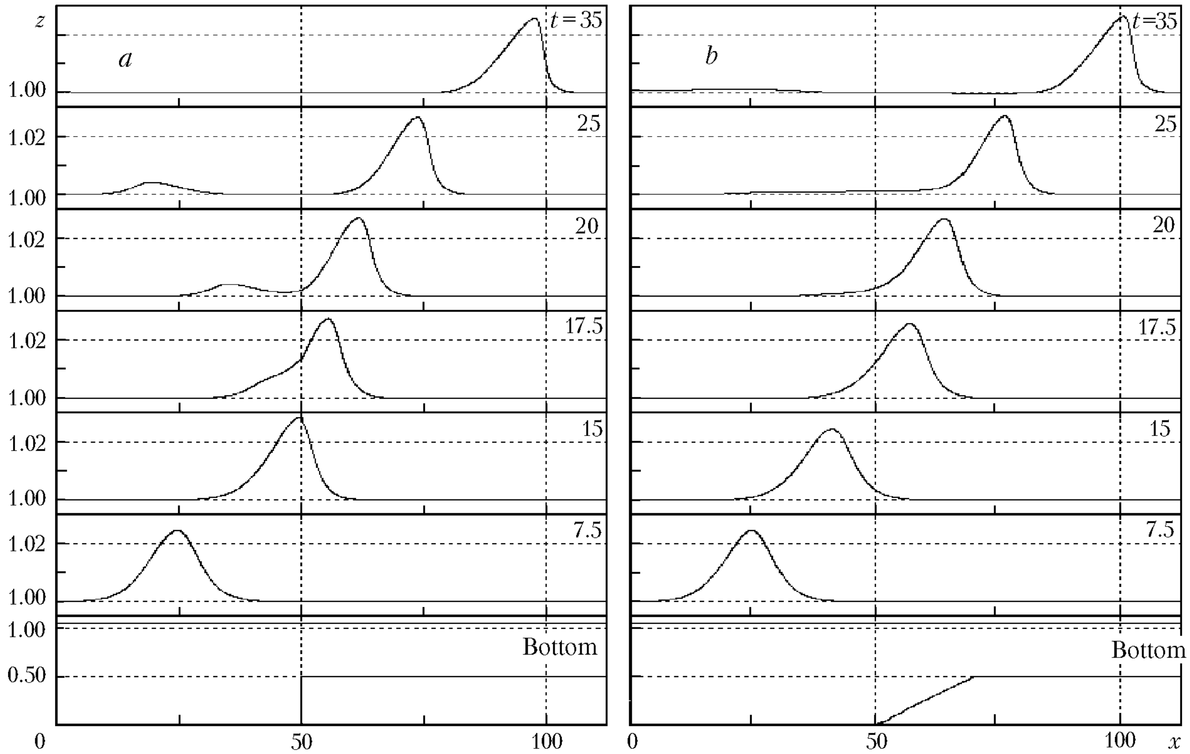


Fig. 2. Change in the wave profile with time in the cases of a rectangular elevation of the bottom (a) and smooth transition (b) from one depth to another. The profile of the bottom is given below. *t*, sec; *z*, *x*, m.

mesh. Simulation with the antidiffusion coefficient beyond this range yields an unstable solution. The reason is that at a small coefficient the diffusion is insufficient, precisely which results in instability, whereas with $\mu$ being high, the diffusion is extremely intense, which also results in instability.

The results of simulation at $\mu$ = 0.39 and 0.49 are presented in Fig. 1. These graphs also show the position of the soliton that was used to interpolate the wave after its formation. This is done for comparison with a "standard" model, so that it could be easier to elucidate the characteristic features in the behavior of the wave described by the system of hydrodynamic equations (1) in the "shallow water" approximation. From the graphs of Fig. 1 it is seen that with time at $\mu$ = 0.39 the soliton wave is transformed into a shock wave and it tends to overturn, i.e., the phenomenon of a gradient catastrophe appears, which is in accord with the theory describing the behavior of the wave in the "shallow water" approximation. When $\mu$ = 0.49, the wave is blurred because of the presence of strong diffusion, but some of its characteristics are described by the model rather well. In both cases, the solution obtained agrees with the soliton in its velocity. If, as the velocity of the wave the velocity of the perturbation maximum is taken, then the difference between the velocities of the soliton and solitary wave obtained numerically is manifested in the third significant figure. As a characteristic of the wave, we may take its area that corresponds to the perturbed mass (the area under the graph of the function $z(x, t)$ minus the area under the free surface). Thus, it was checked whether the numerical model used corresponds to the law of conservation of mass. In our case, the mass is constant accurate to the fourth significant figure at any $\mu$ from the admissible range. In the present work, numerical simulation of the propagation of a solitary wave was performed with the antidiffusion parameter $\mu$ = 0.39.

**Simulation of the Wave with the Bottom of Variable Depth.** To investigate the behavior of the wave upon a sharp change in the depth of the bottom, we performed a series of numerical experiments with different values of the depth: from 10 to 90% of the initial depth of the basin. The results of simulation are presented in Figs. 2 and 3. It is seen that a wave resembling a soliton is reflected from the boundary of the change in the depth, and this wave moves in the opposite direction.

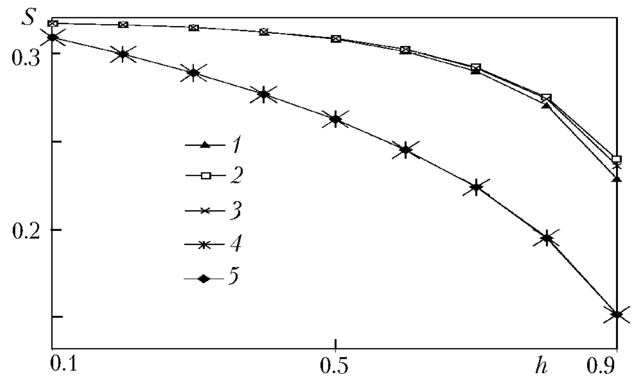


Fig. 3. Area of the wave (quantity of liquid in the wave) vs. the height of the underwater barriers: 1) rectangular crest; 2, 3) trapeziform one with smooth forward and trailing edges, respectively; 4, 5) smooth and rectangular elevation of the bottom level. *S*, m$^2$; *h*, m.

The numerical calculations performed for steps of different heights have shown that the amplitudes of the reflected and transmitted waves depend on the height of the step.

Figure 3 presents the wave area vs. the height of a protrusion. The velocity of the wave downstream of the protrusion is higher than that upstream of it but is lower than the velocity of the soliton used to interpolate the initial wave. The velocity of the soliton is related to the depth [2]. Consequently, this result is quite explicable. It is seen from the diagram that the depth being small, the dependence is linear.

To investigate the behavior of a wave upon smooth change in the depth of the bottom, we also carried out a series of numerical experiments. Passage from one depth to another was smooth, and the length of the slope was equal to half the wavelength. The value of the change in the bottom level was taken to be in the range from 10 to 90% of the initial depth of the basin. These results are presented in Figs. 2 and 3. It is seen that, in contrast to the case of a sharp change in the depth, from the point of the depth drop a wave is reflected whose shape does not resemble a soliton. This wave is much more sloping than a soliton and moves in the opposite direction. Consequently, the area of the transmitted wave depends only on the change in the depth of the bottom and is independent of the length of the smooth elevation.

**Simulation of the Wave Passage over an Underwater Crest.** To investigate the behavior of a wave when it passes a rectangular barrier, we performed a series of numerical experiments with crests of different heights: from 10 to 90% of the depth of the basin with the length of the barrier being equal to half the wavelength. Moreover, we performed numerical experiments with barriers of shape other than a rectangular one, i.e., in the form of a trapezium with sloping left and right boundaries. The results of simulation are presented in Fig. 4.

It is seen from this figure that solitary waves are reflected from both boundaries of this crest, with that reflected from the left boundary (elevation of the level of the bottom) being an ordinary wave whereas that reflected from the right boundary (lowering of the bottom) being an "expansion" wave. Both waves resemble a soliton in shape. After passage of the wave, its velocity is equal to the velocity of the soliton by which it was approximated upstream of the barrier. With smooth edges of the barrier, the same effect is observed as in the case of a change in the depth of the bottom. Reflected waves have a small amplitude and are greatly extended along the *x* axis. This is seen from Fig. 4b.

Figure 3 presents a comparison of changes in the wave area depending on the crest height. The results of simulation with a rectangular step and smooth forward and rear fronts are given. It is seen from the figure that the area of the wave that passed the barriers depends only on the height of the underwater crest and is practically not associated with the smoothness of the slopes. The differences are attributable to the fact that the extension of the barriers with sloping sides are larger than in a rectangular step.

**Simulation of the Collision of Two Waves.** The collision of two solitons is described in [5], where it is shown that initial waves preserve their shape and size after collision, but their phase somewhat changes, i.e., their position after the interaction differs from that they could have had in the absence of interaction. Usually, interactions of two waves moving in the same direction, with one wave overtaking the other, are considered. The formula that deter-

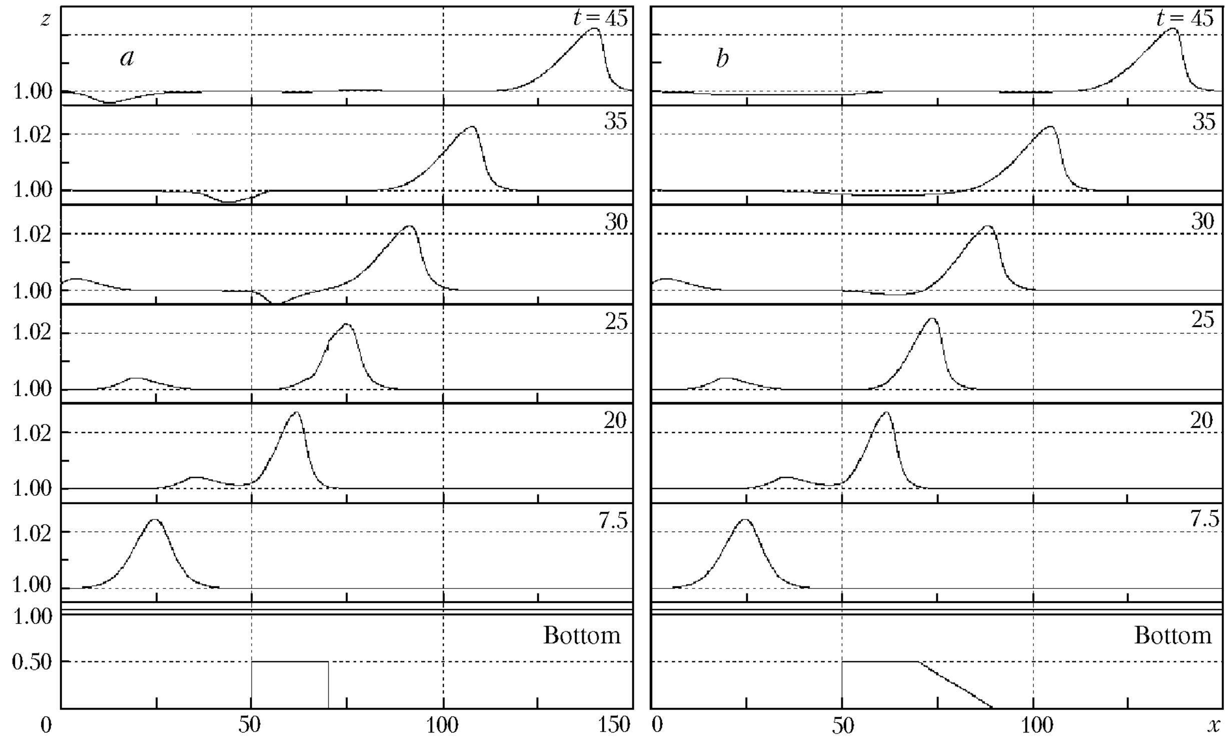


Fig. 4. Change in the wave profile with time in the cases of a rectangular underwater crest (a) and a crest with a smooth trailing edge (b). The profile of the bottom is given below. $t$, sec; $z$, $x$, m.

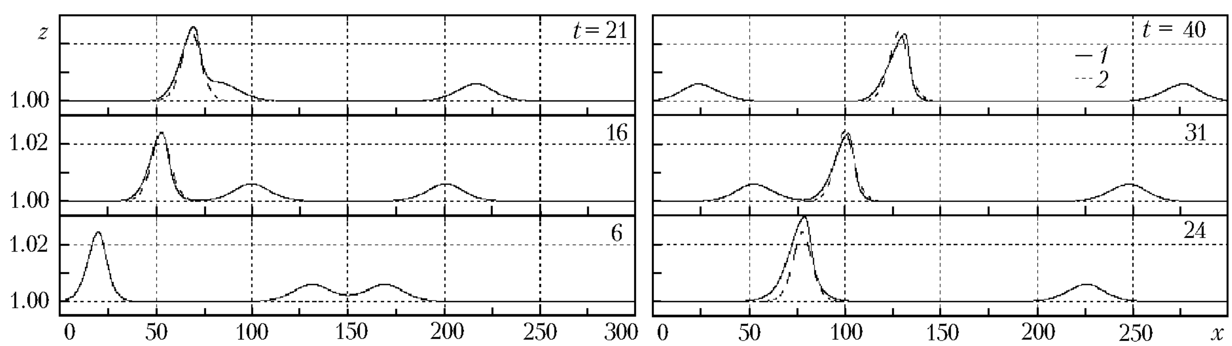


Fig. 5. Change in the wave profile with time upon collision: 1) numerical solution; 2) soliton obtained by solving the Korteweg–de Vries equation. $t$, sec; $z$, $x$, m.

mines the velocity of a solitary wave has the form $v = \sqrt{g(a+h)}$. In the shallow water approximation, the relative velocity of two waves will be low. Therefore, we will limit the discussion to the interaction of two waves moving in opposite directions.

To simulate the collision of two waves, we use the system of equations (1) and difference scheme (4)–(10). We assume that the length of the basin is twice that used in previous calculations. The boundary conditions remain unchanged, whereas the initial ones are used in the form

$$z\,(x,0) = 2k^2\left\{\cosh^{-2}(kx) + \frac{1}{4}\cosh^{-2}\left(\frac{k}{x}(x - L/2)\right)\right\}, \quad u\,(x,0) = 0\,,$$

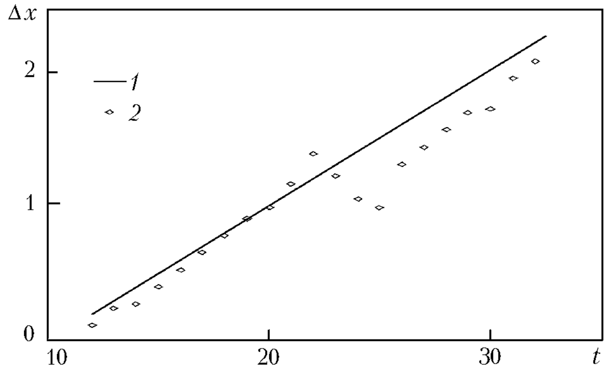


Fig. 6. Time dependence of the difference between the positions of the maxima of the soliton and the wave profile: 1) averaged value for the wave in the absence of collision; 2) calculation points for the case of the collision of waves. $\Delta x$, m; $t$, sec.

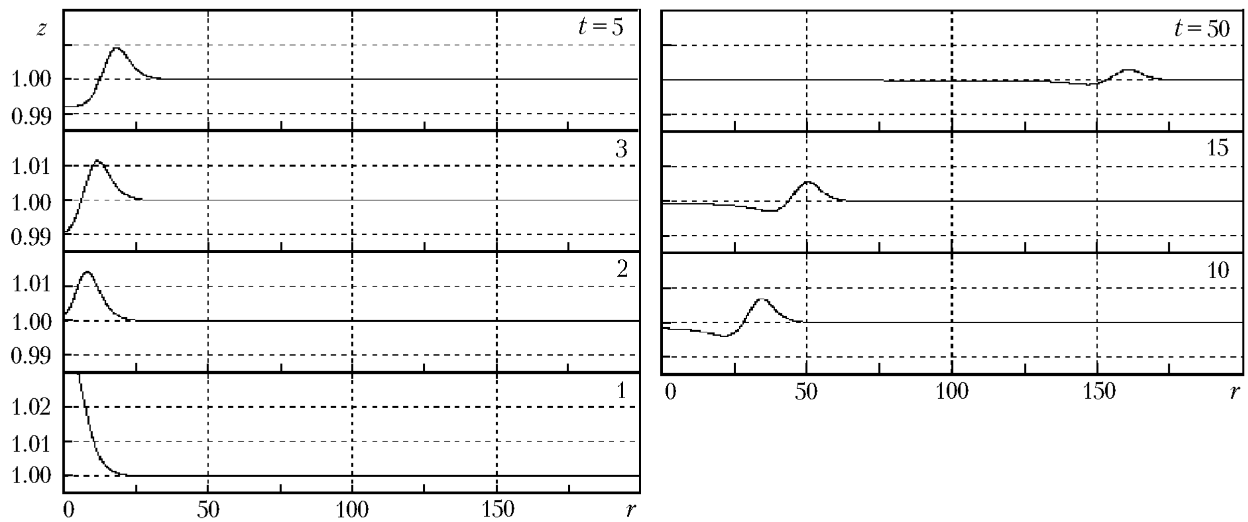


Fig. 7. Change in the wave profile with time in the case of a cylindrical symmetry. $z$, $r$, m; $t$, sec.

$$0 \le x \le L, \quad 0 \le t \le T.$$

Thus, one wave is located at the coordinate origin, and symmetrical boundary conditions are used on the left boundary of the basin. The other wave is at the center of the basin so that the maximum time could pass prior to the arrival of any of the formed waves at the boundary of the basin. This is needed in order not to violate the condition of unperturbance on the right boundary and of symmetricity on the left. Moreover, the first wave after its formation is approximated by a soliton in order that after collision we could compare the shapes of the resulting waves and their phases. The calculations performed showed that, the time of propagation being small, the results for a solitary wave differ insignificantly from those for a soliton.

The results of simulation are presented in Fig. 5. To estimate the change in the phase of the left wave during the experiment, we measured the distance between the maximum of this wave and the maximum of the soliton by which the wave was approximated at the given time instant. The results are presented in Fig. 6. We note that the interaction of the waves occurs at the time from the 16th to the 31st second. For comparison, the figure presents the results obtained in the absence of the right wave, i.e., in the absence of interactions of the waves.

It is seen from the graphs that the shape of the waves is preserved after collision and that after the collision the wave lags in phase behind the noninteracting wave, i.e., the behavior of waves in the shallow water approximation repeats the behavior of solitons.

**Simulation of a Wave in Cylindrical Coordinates.** Above we considered a one-dimensional plane problem, but in real processes one frequently encounters a cylindrical symmetry of the problem that is also one-dimensional. Problems including an axial symmetry occur, for example, when space objects fall into an ocean. We have shown in the foregoing that the initial perturbation is decomposed into two solitons moving in opposite directions. This phenomenon is observed during the decomposition of the initial perturbation that had an axial symmetry and was located at the center. In this case, the system of equations has the form

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + g \frac{\partial z}{\partial r} = 0 \,, \quad \frac{1}{r} \frac{\partial}{\partial r} (ru\,(h + z)) + \frac{\partial z}{\partial t} = 0 \,.$$

This system differs from system (1) only in the second equation, which was supplemented with the term inversely proportional to the distance from the center of symmetry.

The initial and boundary conditions are

$$z\,(r, 0) = 2k^2 \cosh^{-2}(kr) + h_0 \,, \quad u\,(r, 0) = 0 \,, \quad z_r\,(0, t) = 0 \,, \quad u\,(0, t) = 0 \,, \quad 0 \le r \le L \,, \quad 0 \le t \le T \,.$$

The difference scheme is similar to (4)–(10) with allowance for the term added.

The results of simulation are presented in Fig. 7. It is seen that a solitary wave is also formed here. Now, however, the wave consists of two parts: the wave of the elevation of the level and of the subsequent drop in the level, with the amplitude of the rear part being half that of the forward one and its length at the stage of formation being much greater than the length of the wave of elevation (with time they equalize). This structure is also stable, i.e., solitary. The amplitude of the wave decreases substantially with increase in the distance from the center, which is attributable to the cylindrical symmetry of the problem. Its volume is preserved, thus confirming the correction of numerical simulation.

Thus, the data obtained indicate the presence of a wave with soliton properties also in the case of a cylindrical symmetry of the problem.

## CONCLUSIONS

From the numerical experiments carried out, we can draw the following conclusions. In the shallow water approximation, solitary waves move with a constant velocity. Their shape tends asymptotically to a shock wave; therefore it is possible to describe waves in this approximation only up to a certain instant of time. The numerical calculations show that the model is rather satisfactory, which allows one to investigate the influence exerted by the bottom relief on the propagation of gravitational waves in the "shallow water" approximation. It follows from the results obtained that the amplitude of a change in the bottom level plays a decisive role, since it determines the "area" of the transmitted wave. The shape of the transmitted wave is similar to the initial one, with the length of the wave being preserved and the height depending on the barrier. The shape of the transition of the bottom level from one height to another mainly influences the profile of the reflected wave, since the amount of liquid in the latter also depends on the amplitude of the barrier. When the level of the bottom changes jumpwise, the shape of the reflected wave is similar to the initial one.

The results of the computational analysis have shown that propagation of solitary waves can be described by the "shallow water" equations. In this approximation, a solitary wave has the properties typical of a soliton described by the Korteweg–de Vries equation.

## NOTATION

$a$, amplitude of a wave, m; $c_0$, arbitrary constant in the system of Korteweg–de Vries equations; $g$, free-fall acceleration, m$^2$/sec; $h$, step over a coordinate, m; $h(x)$ and $h(r)$, depth of a basin, m; $h_0$, height of the free water surface, m; $h_{xxx}$, third derivative with respect to the coordinate $x$; $h$, arbitrary constant whose square is equal to the amplitude of the wave in the initial condition; $L$, the length of the basin, m; $x$ and $r$, coordinates in the plane and radial directions, m; $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial r}$, and $\frac{\partial}{\partial t}$, derivatives with respect to the coordinates and time; $s$, auxiliary variable; $S$, surface area of the wave, m$^2$; $t$, time, sec; $T$, limiting time of calculation; $u(x, t)$ and $u(r, t)$, components of the velocity vector along the coordinates $x$ and $r$, respectively, m/sec; $u(x, 0)$ and $u(r, 0)$, components of the velocity vector along the axes $x$ and $r$, respectively, at $t = 0$, m/sec; $u_x$, $z_x$, and $(uz)_x$, derivatives with respect to the coordinate $x$; $u_j^i$, grid function of the velocity, m/sec; $v$, velocity of the Korteweg–de Vries solitary wave, m/sec; $z(x, 0)$ and $z(r, 0)$, vertical coordinate of the water surface at the initial instant of time, m; $z(x, t)$ and $z(r, t)$, vertical coordinate of the water surface, m; $z_x(0, t)$, derivatives of $z(x, t)$ and $z(r, t)$ at $x = 0$ and $r = 0$; $z_t$, derivative with respect to time; $\tilde{z}_j$, auxiliary grid function; $z_j^i$, grid function corresponding to the vertical coordinate of the water surface (the subscript denotes the number of the point over $x$, the superscript — the number of the layer in the time $t$); $\Phi_{j+1/2}$ and $\tilde{\Phi}_{j+1/2}$, calculated and correlated antidiffusion fluxes; $\varepsilon_{j+1/2}$, proportionality factor in a transport flow [4]; $\mu$, antidiffusion coefficient; $\nu_{j+1/2}$, proportionality factor in a diffusion flow [4].

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